

CONFORMAL SEMI-INVARIANT SUBMERSIONS

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ABSTRACT. As a generalization of semi-invariant submersions, we introduce conformal semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds. We give examples, investigate the geometry of foliations which are arisen from the definition of a conformal submersion and show that there are certain product structures on the total space of a conformal semi-invariant submersion. Moreover, we also check the harmonicity of such submersions and find necessary and sufficient conditions of a conformal semi-invariant submersion to be totally geodesic.

1. INTRODUCTION

One of the main method to compare two manifolds and transfer certain structures from a manifold to another manifold is to define appropriate smooth maps between them. Given two manifolds, if the rank of a differential map is equal to the dimension of the source manifold, then such maps are called immersions and if the rank of a differential map is equal to the target manifold, then such maps are called submersions. Moreover, if these maps are isometry between manifolds, then the immersion is called isometric immersion (Riemannian submanifold) and the submersion is called Riemannian submersion. Riemannian submersions between Riemannian manifolds were studied by O'Neill [21] and Gray [14], for recent developments on the geometry of Riemannian submanifolds and Riemannian submersions, see: [5] and [13], respectively.

The theory of submanifolds of Kähler manifolds is one of the important branches of differential geometry. A submanifold of a Kähler manifold is a complex (invariant) submanifold if the tangent space of the submanifold at each point is invariant with respect to the almost complex structure of the Kähler manifold. Besides complex submanifolds of a Kähler manifold, there is another important class of submanifolds called totally real submanifolds. A totally real submanifold of a Kähler manifold \bar{M} is a submanifold of \bar{M} such that the almost complex structure J of \bar{M} carries the tangent space of the submanifold at each point into its normal space and the main properties of such submanifolds established in [6], [20] and [31]. On the other hand, CR-submanifolds were defined by Bejancu [3] as a generalization of complex and totally real submanifolds. A CR-submanifold is called proper if it is neither complex nor totally real submanifold. Many authors have studied above real submanifolds in various ambient manifolds and many interesting results were obtained, see ([5], page: 322) for a survey on all these results.

As analogue of holomorphic submanifolds, holomorphic submersions were introduced by Watson [29] in seventies by using the notion of almost complex map. We note that almost Hermitian submersions have been extended to the almost contact manifolds [7], locally conformal Kähler manifolds [19], quaternion Kähler manifolds [16] and Paraquaternionic manifold [4], [28], see [13] for holomorphic submersions and their extensions to the other manifolds. The main property of such maps is that their vertical distributions are invariant with respect to almost complex map of total space. Therefore, the second author of the present paper considered a new submersion defined on an almost Hermitian manifold such that the vertical distribution is anti-invariant with respect to almost complex structure [25]. He showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. This new class of submersions

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which is called anti-invariant submersions can be seen as an analogue of totally real submanifolds in the submersion theory. As a generalization of holomorphic submersions and anti-invariant submersions, the second author introduced semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds and then he studied the geometry of such maps in [26]. We recall that a Riemannian submersion F from an almost Hermitian manifold (M, J_M, g_M) with almost complex structure J_M to a Riemannian manifold (N, g_N) is called a semi-invariant submersion if the fibers have differentiable distributions D and D^\perp such that D is invariant with respect to J_M and its orthogonal complement D^\perp is totally real distribution, i.e., $J_M(D_p^\perp) \subseteq (\ker F_*)^\perp$. Obviously, almost Hermitian submersions [29] and anti-invariant submersions [25] are semi-invariant submersions with $D^\perp = \{0\}$ and $D = \{0\}$, respectively. These new submersions have been studied in different total spaces, see: [11], [18], [22], [23], [24], [27].

On the other hand, as a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [2]: Suppose that (M, g_M) and (B, g_B) are Riemannian manifolds and $F : M \rightarrow B$ is a smooth submersion, then F is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(X, Y) = g_B(F_*X, F_*Y)$$

for every $X, Y \in \Gamma((\ker F_*)^\perp)$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [12] and Ishihara [17]. We also note that a horizontally conformal submersion $F : M \rightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$(1.1) \quad \mathcal{H}(\text{grad} \lambda) = 0$$

at $p \in M$, where \mathcal{H} is the projection on the horizontal space $(\ker F_*)^\perp$. One can see that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps does not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [15]. They obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism, see also [8], [9] and [10] for the harmonicity of conformal holomorphic submersions. Moreover, in [1], we introduce conformal anti-invariant submersions, give examples and investigate the geometry of such submersions.

In this paper, we study conformal semi-invariant submersions as a generalization of semi-invariant submersions and investigate the geometry of the total space and the base space for the existence of such submersions.

The paper is organized as follows. In the second section, we gather main notions and formulas for other sections. In section 3, we introduce conformal semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds, give examples and investigate the geometry of leaves of the horizontal distribution and the vertical distribution. In this section we also show that there are certain product structures on the total space of a conformal semi-invariant submersion. In section 4, we find necessary and sufficient conditions for a conformal semi-invariant submersion to be harmonic and totally geodesic, respectively.

2. PRELIMINARIES

In this section, we define almost Hermitian manifolds, recall the notion of (horizontally) conformal submersions between Riemannian manifolds and give a brief review of basic facts of (horizontally) conformal submersions. Let (M, g_M) be an almost Hermitian manifold. This means [32] that M admits a tensor field J of type (1,1) on M such that, $\forall X, Y \in \Gamma(TM)$, we have

$$(2.1) \quad J^2 = -I, \quad g_M(X, Y) = g_M(JX, JY).$$

An almost Hermitian manifold M is called Kähler manifold if

$$(2.2) \quad (\nabla_X^M J)Y = 0, \quad \forall X, Y \in \Gamma(TM),$$

where ∇^M is the Levi-Civita connection on M . Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

Definition 2.1. ([2]) *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called horizontally weakly conformal or semi conformal at x if either*

- (i) $d\varphi_x = 0$, or
- (ii) $d\varphi_x$ maps the horizontal space $\mathcal{H}_x = (\ker(d\varphi_x))^\perp$ conformally onto $T_{\varphi_*}N$, i.e., $d\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that

$$(2.3) \quad h(d\varphi_x X, d\varphi_x Y) = \Lambda(x)g(X, Y) \quad (X, Y \in \mathcal{H}_x).$$

Note that we can write the last equation more succinctly as

$$(\varphi^*h)_x|_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x)g_x|_{\mathcal{H}_x \times \mathcal{H}_x}.$$

With the above definition of critical point, a point x is of type (i) in Definition 2.1 if and only if it is a critical point of φ ; we shall call a point of type (ii) a *regular point*. At a critical point, $d\varphi_x$ has rank 0; at a regular point, $d\varphi_x$ has rank n and φ is submersion. The number $\Lambda(x)$ is called the *square dilation* (of φ at x); it is necessarily non-negative; its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the *dilation* (of φ at x). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M . It is clear that if φ has no critical points, then we call it a (*horizontally*) conformal submersion.

Next, we recall the following definition from [2]. Let $F : M \longrightarrow N$ be a submersion. A vector field E on M is said to be projectable if there exists a vector field \tilde{E} on N , such that $F_*(E_x) = \tilde{E}_{F(x)}$ for all $x \in M$. In this case E and \tilde{E} are called F -related. A horizontal vector field Y on (M, g) is called basic, if it is projectable. It is well known fact, that is \tilde{Z} is a vector field on N , then there exists a unique basic vector field Z on M , such that Z and \tilde{Z} are F -related. The vector field Z is called the horizontal lift of \tilde{Z} .

The fundamental tensors of a submersion were introduced in [21]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A defined for vector fields E, F on M by

$$(2.4) \quad A_E F = \mathcal{V}\nabla_{\mathcal{H}E}^M \mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}^M \mathcal{V}F$$

$$(2.5) \quad T_E F = \mathcal{H}\nabla_{\mathcal{V}E}^M \mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}^M \mathcal{H}F$$

where \mathcal{V} and \mathcal{H} are the vertical and horizontal projections (see [13]). On the other hand, from (2.4) and (2.5), we have

$$(2.6) \quad \nabla_V^M W = T_V W + \hat{\nabla}_V W$$

$$(2.7) \quad \nabla_V^M X = \mathcal{H}\nabla_V^M X + T_V X$$

$$(2.8) \quad \nabla_X^M V = A_X V + \mathcal{V}\nabla_X^M V$$

$$(2.9) \quad \nabla_X^M Y = \mathcal{H}\nabla_X^M Y + A_X Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V^M W$. If X is basic, then $\mathcal{H}\nabla_V^M X = A_X V$. It is easily seen that for $x \in M$, $X \in \mathcal{H}_x$ and $V \in \mathcal{V}_x$ the linear operators $T_V, A_X : T_x M \longrightarrow T_x M$ are skew-symmetric, that is

$$-g(T_V E, G) = g(E, T_V G) \quad \text{and} \quad -g(A_X E, G) = g(E, A_X G)$$

for all $E, G \in T_x M$. We also see that the restriction of T to the vertical distribution $T|_{V \times V}$ is exactly the second fundamental form of the fibres of F . Since T_V skew-symmetric we get: F has totally geodesic fibres if and only if $T \equiv 0$.

We now recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \rightarrow N$ is a smooth map between them. Then the differential of φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^*TN) \rightarrow M$, where φ^*TN is the pullback bundle which has fibres $(\varphi^*TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^*TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(2.10) \quad (\nabla \varphi_*)(X, Y) = \nabla_X^\varphi \varphi_*(Y) - \varphi_*(\nabla_X^M Y)$$

for $X, Y \in \Gamma(TM)$, where ∇^φ is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \rightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla \varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^*TN)$ defined by

$$(2.11) \quad \tau(\varphi) = div \varphi_* = \sum_{i=1}^m (\nabla \varphi_*)(e_i, e_i),$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame on M . Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, for details, see [2]. Finally, we recall the following lemma from [2].

Lemma 2.1. *Suppose that $F : M \rightarrow N$ is a horizontally conformal submersion. Then, for any horizontal vector fields X, Y and vertical fields V, W we have*

- (i) $(\nabla F_*)(X, Y) = X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) - g(X, Y)F_*(grad \ln \lambda);$
- (ii) $(\nabla F_*)(V, W) = -F_*(T_V W);$
- (iii) $(\nabla F_*)(X, V) = -F_*(\nabla_X^M V) = -F_*(A_X V).$

3. CONFORMAL SEMI-INVARIANT SUBMERSIONS

In this section, we define conformal semi-invariant submersions from an almost Hermitian manifold onto a Riemannian manifold, investigate the integrability of distributions and show that there are certain product structures on the total space of such submersions.

Definition 3.1. *Let M be a complex m -dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N . A horizontally conformal submersion $F : M \rightarrow N$ with dilation λ is called conformal semi-invariant submersion if there is a distribution $D_1 \subseteq \ker F_*$ such that*

$$(3.1) \quad \ker F_* = D_1 \oplus D_2$$

and

$$(3.2) \quad J(D_1) = D_1, J(D_2) \subseteq (\ker F_*)^\perp,$$

where D_2 is orthogonal complementary to D_1 in $\ker F_*$.

We note that it is known that the distribution $\ker F_*$ is integrable. Hence, above definition implies that the integral manifold (fiber) $F^{-1}(q)$, $q \in N$, of $\ker F_*$ is CR-submanifold of M . For CR-submanifolds, see [3] and [5]. We now give some examples of conformal semi-invariant submersions.

Example 3.1. *Every semi-invariant submersion from an almost Hermitian manifold to a Riemannian manifold is a conformal semi-invariant submersion with $\lambda = I$, where I denotes the identity function.*

We say that a conformal semi-invariant submersion is proper if $\lambda \neq I$. We now present an example of a proper conformal semi-invariant submersion. In the following R^{2m} denotes the Euclidean $2m$ -space with the standard metric. An almost complex structure J on R^{2m} is said to be compatible if (R^{2m}, J) is complex analytically isometric to the complex number space C^m with

the standard flat Kählerian metric. We denote by J the compatible almost complex structure on R^{2m} defined by

$$J(a^1, \dots, a^{2m}) = (-a^2, a^1, \dots, -a^{2m}, a^{2m-1}).$$

Example 3.2. Let F be a submersion defined by

$$F : \begin{array}{ccc} R^6 & \longrightarrow & R^2 \\ (x_1, x_2, x_3, x_4, x_5, x_6) & \longrightarrow & (e^{x_3} \cos x_5, e^{x_3} \sin x_5), \end{array}$$

where $x_5 \in \mathbb{R} - \{k\frac{\pi}{2}, k\pi\}$, $k \in \mathbb{R}$. Then it follows that

$$\ker F_* = \text{span}\{V_1 = \partial x_1, V_2 = \partial x_2, V_3 = \partial x_4, V_4 = \partial x_6\}$$

and

$$(\ker F_*)^\perp = \text{span}\{X_1 = e^{x_3} \cos x_5 \partial x_3 - e^{x_3} \sin x_5 \partial x_5, X_2 = e^{x_3} \sin x_5 \partial x_3 + e^{x_3} \cos x_5 \partial x_5\}.$$

Hence we have $JV_1 = V_2$, $JV_3 = -e^{-x_3} \cos x_5 X_1 - e^{-x_3} \sin x_5 X_2$ and $JV_4 = e^{-x_3} \sin x_5 X_1 - e^{-x_3} \cos x_5 X_2$. Thus it follows that $D_1 = \text{span}\{V_1, V_2\}$ and $D_2 = \text{span}\{V_3, V_4\}$. Also by direct computations, we get

$$F_* X_1 = (e^{x_3})^2 \partial y_1, F_* X_2 = (e^{x_3})^2 \partial y_2.$$

Hence, we have

$$g_2(F_* X_1, F_* X_1) = (e^{x_3})^2 g_1(X_1, X_1), \quad g_2(F_* X_2, F_* X_2) = (e^{x_3})^2 g_1(X_2, X_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of R^6 and R^2 . Thus F is a conformal semi-invariant submersion with $\lambda = e^{x_3}$.

We now investigate the integrability of the distributions D_1 and D_2 .

Lemma 3.1. Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) . Then

(i) The distribution D_2 is always integrable.

(ii) The distribution D_1 integrable if and only if $(\nabla F_*)(Y, JX) - (\nabla F_*)(X, JY) \in \Gamma(F_*(\mu))$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Since the fibers of conformal semi-invariant submersions from Kähler manifolds are CR-submanifolds and T is the second fundamental form of the fibers, (i) can be deduced from Theorem 1.1 of [[3], p.39]. (ii) We note that the distribution D_1 integrable if and only if $g_M([X, Y], Z) = g_M([X, Y], W) = 0$ for $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $W \in \Gamma((\ker F_*)^\perp)$. Since $\ker F_*$ is integrable, we immediately have $g_M([X, Y], W) = 0$. Thus D_1 is integrable if and only if $g_M([X, Y], Z) = 0$. Since F is a conformal submersion, by using (2.1) and Lemma 2.1 we have

$$g_M([X, Y], Z) = \frac{1}{\lambda^2} g_N(F_*(\nabla_X^M JY), F_* JZ) - \frac{1}{\lambda^2} g_N(F_*(\nabla_Y^M JX), F_* JZ).$$

Then using (2.10) we get

$$g_M([X, Y], Z) = \frac{1}{\lambda^2} g_N((\nabla F_*)(Y, JX) - (\nabla F_*)(X, JY), F_* JZ).$$

Thus proof is complete. \square

Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) . We denote the complementary distribution to JD_2 in $(\ker F_*)^\perp$ by μ . Then for $V \in \Gamma(\ker F_*)$, we write

$$(3.3) \quad JV = \phi V + \omega V$$

where $\phi V \in \Gamma(D_1)$ and $\omega V \in \Gamma(JD_2)$. Also for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$(3.4) \quad JX = \mathcal{B}X + \mathcal{C}X,$$

where $\mathcal{B}X \in \Gamma(D_2)$ and $\mathcal{C}X \in \Gamma(\mu)$. Then by using (3.3), (3.4), (2.6) and (2.7) we get

$$(3.5) \quad (\nabla_V^M \phi)W = \mathcal{B}T_V W - T_V \omega W$$

$$(3.6) \quad (\nabla_V^M \omega)W = \mathcal{C}T_V W - T_V \phi W$$

for $V, W \in \Gamma(\ker F_*)$, where

$$(\nabla_V^M \phi)W = \hat{\nabla}_V \phi W - \phi \hat{\nabla}_V W$$

and

$$(\nabla_V^M \omega)W = \mathcal{H}\nabla_V^M \omega W - \omega \hat{\nabla}_V W.$$

We now study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 3.1. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ is integrable if and only if*

$$A_Y \omega \mathcal{B}X - A_X \omega \mathcal{B}Y + J(A_Y \mathcal{C}X - A_X \mathcal{C}Y) \notin \Gamma(D_1)$$

and

$$\begin{aligned} \lambda^2 g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* JW) &= g_M(A_Y \mathcal{B}X - A_X \mathcal{B}Y - \mathcal{C}Y(\ln \lambda)X + \mathcal{C}X(\ln \lambda)Y \\ &\quad + 2g_M(X, \mathcal{C}Y) \operatorname{grad} \ln \lambda, JW) \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$.

Proof. The distribution $(\ker F_*)^\perp$ is integrable on M if and only if

$$g_M([X, Y], V) = 0 \quad \text{and} \quad g_M([X, Y], W) = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$. Using (2.1), (2.2) and (3.4), we get

$$g_M([X, Y], V) = g_M(\nabla_X^M \mathcal{B}Y, JV) + g_M(\nabla_X^M \mathcal{C}Y, JV) - g_M(\nabla_Y^M \mathcal{B}X, JV) - g_M(\nabla_Y^M \mathcal{C}X, JV).$$

Also using (2.9), we get

$$g_M([X, Y], V) = -g_M(\mathcal{B}Y, \nabla_X^M JV) + g_M(A_X \mathcal{C}Y, JV) + g_M(\mathcal{B}X, \nabla_Y^M JV) - g_M(A_Y \mathcal{C}X, JV).$$

From (2.2), (2.8) and (3.3), we derive

$$(3.7) \quad g_M([X, Y], V) = g_M(A_Y \omega \mathcal{B}X - A_X \omega \mathcal{B}Y - J A_X \mathcal{C}Y + J A_Y \mathcal{C}X, V).$$

On the other hand, from (2.1), (2.2) and (3.4) we derive

$$g_M([X, Y], W) = g_M(\nabla_X^M \mathcal{B}Y, JW) + g_M(\nabla_X^M \mathcal{C}Y, JW) - g_M(\nabla_Y^M \mathcal{B}X, JW) - g_M(\nabla_Y^M \mathcal{C}X, JW).$$

Since F is a conformal submersion, using (2.10) and Lemma 2.1 we arrive at

$$\begin{aligned} g_M([X, Y], W) &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(X, \mathcal{B}Y), F_* JW) + \frac{1}{\lambda^2} g_N((\nabla F_*)(Y, \mathcal{B}X), F_* JW) \\ &\quad + \frac{1}{\lambda^2} g_N\{-X(\ln \lambda)F_* \mathcal{C}Y - \mathcal{C}Y(\ln \lambda)F_* X + g_M(X, \mathcal{C}Y)F_*(\operatorname{grad} \ln \lambda) + \nabla_X^F F_* \mathcal{C}Y, F_* JW\} \\ &\quad - \frac{1}{\lambda^2} g_N\{-Y(\ln \lambda)F_* \mathcal{C}X - \mathcal{C}X(\ln \lambda)F_* Y + g_M(Y, \mathcal{C}X)F_*(\operatorname{grad} \ln \lambda) + \nabla_Y^F F_* \mathcal{C}X, F_* JW\}. \end{aligned}$$

Thus from (2.10) and (2.8) we have

$$\begin{aligned} g_M([X, Y], W) &= -g_M(A_X \mathcal{B}Y, JW) + g_M(A_Y \mathcal{B}X, JW) - \frac{1}{\lambda^2} g_M(\operatorname{grad} \ln \lambda, X) g_N(F_* \mathcal{C}Y, F_* JW) \\ &\quad - \frac{1}{\lambda^2} g_M(\operatorname{grad} \ln \lambda, \mathcal{C}Y) g_N(F_* X, F_* JW) + \frac{1}{\lambda^2} g_M(X, \mathcal{C}Y) g_N(F_*(\operatorname{grad} \ln \lambda), F_* JW) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_X^F F_* \mathcal{C}Y, F_* JW) + \frac{1}{\lambda^2} g_M(\operatorname{grad} \ln \lambda, Y) g_N(F_* \mathcal{C}X, F_* JW) \\ &\quad + \frac{1}{\lambda^2} g_M(\operatorname{grad} \ln \lambda, \mathcal{C}X) g_N(F_* Y, F_* JW) - \frac{1}{\lambda^2} g_M(Y, \mathcal{C}X) g_N(F_*(\operatorname{grad} \ln \lambda), F_* JW) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla_Y^F F_* \mathcal{C}X, F_* JW). \end{aligned}$$

Moreover, using Definition 3.1, we obtain

$$(3.8) \quad g_M([X, Y], W) = g_M(A_Y \mathcal{B}X - A_X \mathcal{B}Y - \mathcal{C}Y(\ln \lambda)X + \mathcal{C}X(\ln \lambda)Y + 2g_M(X, \mathcal{C}Y)grad \ln \lambda, JW) - \frac{1}{\lambda^2} g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* JW).$$

Thus proof follows from (3.7) and (3.8). \square

Next theorem gives a necessary and sufficient condition for conformal submersion to be a homothetic map.

Theorem 3.2. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with integrable distribution $(\ker F_*)^\perp$. Then F is a horizontally homothetic map if and only if*

$$(3.9) \quad \lambda^2 g_M(A_Y \mathcal{B}X - A_X \mathcal{B}Y, JW) = g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* JW)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $W \in \Gamma(D_2)$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $W \in \Gamma(D_2)$, from (3.8) we have

$$g_M([X, Y], W) = g_M(A_Y \mathcal{B}X - A_X \mathcal{B}Y - \mathcal{C}Y(\ln \lambda)X + \mathcal{C}X(\ln \lambda)Y + 2g_M(X, \mathcal{C}Y)grad \ln \lambda, JW) - \frac{1}{\lambda^2} g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* JW).$$

If F is a horizontally homothetic map then we get (3.9). Conversely, if (3.9) is satisfied then we get

$$(3.10) \quad g_M(-g_M(grad \ln \lambda, \mathcal{C}Y)X + g_M(grad \ln \lambda, \mathcal{C}X)Y + 2g_M(X, \mathcal{C}Y)grad \ln \lambda, JW) = 0.$$

Now, taking $Y = JW$ for $W \in \Gamma(D_2)$ in (3.10), we have $g_M(grad \ln \lambda, \mathcal{C}X)g_M(JW, JW) = 0$. Thus, λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = \mathcal{C}X$ for $X \in \Gamma(\mu)$ in (3.10) we obtain

$$2g_M(X, \mathcal{C}^2 X)g_M(grad \ln \lambda, JW) = 2g_M(X, X)g_M(grad \ln \lambda, JW) = 0.$$

From above equation, λ is a constant on $\Gamma(JD_2)$. This completes the proof. \square

As conformal version of anti-holomorphic semi-invariant submersion ([30]), a conformal semi-invariant submersion is called a conformal anti-holomorphic semi-invariant submersion if $J(D_2) = (\ker F_*)^\perp$. For a conformal anti-holomorphic semi-invariant submersion, from Theorem 3.1 we have the following.

Corollary 3.1. *Let F be a conformal anti-holomorphic semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the following assertions are equivalent to each other;*

- (i) $(\ker F_*)^\perp$ is integrable
- (ii) $g_N(F_* JW_1, (\nabla F_*)(V, JW_2)) = g_N(F_* JW_2, (\nabla F_*)(V, JW_1))$

for $W_1, W_2 \in \Gamma(D_2)$ and $V \in \Gamma(\ker F_*)$.

For the geometry of leaves of the horizontal distribution, we have the following theorem.

Theorem 3.3. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$A_X \mathcal{C}Y + \mathcal{V} \nabla_X^M \mathcal{B}Y \in \Gamma(D_2)$$

and

$$\lambda^2 \{g_M(A_X \mathcal{B}Y - \mathcal{C}Y(\ln \lambda)X + g_M(X, \mathcal{C}Y)grad \ln \lambda, JW)\} = g_N(\nabla_X^F F_* JW, F_* \mathcal{C}Y)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$.

Proof. The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M if and only if $g_M(\nabla_X^M Y, V) = 0$ and $g_M(\nabla_X^M Y, W) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$. Then by using (2.1), (2.2), (2.8), (2.9) and (3.4), we get

$$(3.11) \quad g_M(\nabla_X^M Y, V) = -g_M(\phi(A_X CY + \mathcal{V}\nabla_X^M \mathcal{B}Y), V).$$

On the other hand, from (2.1), (2.2) and (3.4) we get

$$g_M(\nabla_X^M Y, W) = -g_M(\mathcal{B}Y, \nabla_X^M JW) - g_M(CY, \nabla_X^M JW).$$

Since F is a conformal submersion, using (2.9), (2.10) and Lemma 2.1 we arrive at

$$\begin{aligned} g_M(\nabla_X^M Y, W) &= -g_M(\mathcal{B}Y, A_X JW) + \frac{1}{\lambda^2} g_M(\text{grad} \ln \lambda, JW) g_N(F_* X, F_* CY) \\ &\quad - \frac{1}{\lambda^2} g_M(X, JW) g_N(F_*(\text{grad} \ln \lambda), F_* CY) - \frac{1}{\lambda^2} g_N(\nabla_X^F F_* JW, F_* CY). \end{aligned}$$

Conformal semi-invariant F implies that

$$\begin{aligned} g_M(\nabla_X^M Y, V) &= g_M(A_X \mathcal{B}Y - CY(\ln \lambda)X + g_M(X, CY) \text{grad} \ln \lambda, JW) \\ (3.12) \quad &\quad - \frac{1}{\lambda^2} g_N(\nabla_X^F F_* JW, F_* CY). \end{aligned}$$

Thus proof follows from (3.11) and (3.12). \square

Next we give new conditions for conformal semi-invariant submersions to be horizontally homothetic map. But we first give the following definition.

Definition 3.2. Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then we say that D_2 is parallel along $(\ker F_*)^\perp$ if $\nabla_X^M W \in \Gamma(D_2)$ for $X \in \Gamma((\ker F_*)^\perp)$ and $W \in \Gamma(D_2)$.

Corollary 3.2. Let $F : (M, g_M, J) \rightarrow (N, g_N)$ be a conformal semi-invariant submersion such that D_2 is parallel along $(\ker F_*)^\perp$. Then F is a horizontally homothetic map if and only if

$$(3.13) \quad \lambda^2 g_M(A_X \mathcal{B}Y, JW) = g_N(\nabla_X^F F_* JW, F_* CY)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $W \in \Gamma(D_2)$.

Proof. (3.13) implies that

$$(3.14) \quad -g_M(\text{grad} \ln \lambda, CY) g_M(X, JW) + g_M(X, CY) g_M(\text{grad} \ln \lambda, JW) = 0.$$

Now, taking $X = JW$ for $W \in \Gamma(D_2)$ in (3.14), we get

$$g_M(\text{grad} \ln \lambda, CY) g_M(JW, JW) = 0.$$

Thus, λ is a constant on $\Gamma(\mu)$. On the other hand, taking $X = CY$ for $Y \in \Gamma(\mu)$ in (3.14) we derive

$$g_M(CY, CY) g_M(\text{grad} \ln \lambda, JW) = 0.$$

From above equation, λ is a constant on $\Gamma(JD_2)$. The converse is clear from (3.12). \square

In particular, if F is a conformal anti-holomorphic semi-invariant submersion, then we have the following.

Corollary 3.3. Let F be a conformal anti-holomorphic semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the following assertions are equivalent to each other;

- (i) $(\ker F_*)^\perp$ defines a totally geodesic foliation on M .
- (ii) $(\nabla F_*)(V, JW_1) \in \Gamma(F_*(\mu))$

for $W_1, W_2 \in \Gamma(D_2)$ and $V \in \Gamma(\ker F_*)$.

In the sequel we are going to investigate the geometry of leaves of the distribution $\ker F_*$.

Theorem 3.4. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)$ defines a totally geodesic foliation on M if and only if*

$$\lambda^2 \{g_M(\mathcal{C}T_U \phi V + A_{\omega V} \phi U + g_M(\omega V, \omega U) \text{grad} \ln \lambda, X)\} = g_N(\nabla_{\omega V}^F F_* X, F_* \omega U)$$

and

$$T_V \omega U + \hat{\nabla}_V \phi U \in \Gamma(D_1)$$

for $U, V \in \Gamma(\ker F_*)$, $X \in \Gamma(\mu)$ and $W \in \Gamma(D_2)$.

Proof. The distribution $(\ker F_*)$ defines a totally geodesic foliation on M if and only if $g_M(\nabla_U^M V, X) = 0$ and $g_M(\nabla_U^M V, JW) = 0$ for $U, V \in \Gamma(\ker F_*)$, $X \in \Gamma(\mu)$ and $W \in \Gamma(D_2)$. Using (2.1), (2.2) and (3.3), we have

$$g_M(\nabla_U^M V, X) = g_M(\nabla_U^M \phi V, JX) + g_M(\phi U, \nabla_{\omega V}^M X) + g_M(\omega U, \nabla_{\omega V}^M X).$$

Since F is a conformal submersion, from (2.6), (2.9), (2.10) and Lemma 2.1 we arrive at

$$\begin{aligned} g_M(\nabla_U^M V, X) &= g_M(T_U \phi V, JX) + g_M(\phi U, A_{\omega V} X) - \frac{1}{\lambda^2} g_M(\text{grad} \ln \lambda, X) g_N(F_* \omega V, F_* \omega U) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_{\omega V}^F F_* X, F_* \omega U). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} g_M(\nabla_U^M V, X) &= g_M(-\mathcal{C}T_U \phi V - A_{\omega V} \phi U - g_M(\omega V, \omega U) \text{grad} \ln \lambda, X) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_{\omega V}^F F_* X, F_* \omega U). \end{aligned} \tag{3.15}$$

On the other hand, by using (2.1), (2.2) and (3.3), we get

$$g_M(\nabla_U^M V, JW) = -g_M(\omega(\hat{\nabla}_V \phi U + T_V \omega U), JW). \tag{3.16}$$

Thus proof follows from (3.15) and (3.16). \square

Next we give certain conditions for dilation λ to be constant on μ . We first give the following definition.

Definition 3.3. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then we say that μ is parallel along $\ker F_*$ if $\nabla_U^M X \in \Gamma(\mu)$ for $X \in \Gamma(\mu)$ and $U \in \Gamma(\ker F_*)$.*

Corollary 3.4. *Let $F : (M, g_M, J) \rightarrow (N, g_N)$ be a conformal semi-invariant submersion such that μ is parallel along $(\ker F_*)$. Then F is a constant on μ if and only if*

$$\lambda^2 g_M(\mathcal{C}T_U \phi V + A_{\omega V} \phi U, X) = g_N(\nabla_{\omega V}^F F_* X, F_* \omega U) \tag{3.17}$$

for $U, V \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$.

Proof. If we have (3.17) then we have

$$g_M(\omega V, \omega U) g_M(\text{grad} \ln \lambda, X) = 0. \tag{3.18}$$

From above equation, λ is a constant on $\Gamma(\mu)$. The converse comes from (3.15) \square

From Theorem 3.3 and Theorem 3.4, we have the following decomposition for total space;

Theorem 3.5. *Let $F : (M, g_M, J) \rightarrow (N, g_N)$ be a conformal semi-invariant submersion, where (M, g_M, J) is a Kähler manifold (N, g_N) is a Riemannian manifold. Then M is a locally product manifold of the form $M_{(\ker F_*)} \times_\lambda M_{(\ker F_*)^\perp}$ if and only if*

$$\begin{aligned} \lambda^2 \{g_M(\mathcal{C}T_U \phi V + A_{\omega V} \phi U + g_M(\omega V, \omega U) \text{grad} \ln \lambda, X)\} &= g_N(\nabla_{\omega V}^F F_* X, F_* \omega U), \\ T_V \omega U + \hat{\nabla}_V \phi U &\in \Gamma(D_1) \end{aligned}$$

and

$$A_X CY + \mathcal{V} \nabla_X^M \mathcal{B}Y \in \Gamma(D_2),$$

$$\lambda^2 \{g_M(A_X \mathcal{B}Y - CY(\ln \lambda)X + g_M(X, CY) \text{grad} \ln \lambda, JW)\} = g_N(\nabla_X^F F_* JW, F_* CY)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V, W \in \Gamma(\ker F_*)$.

Since $(\ker F_*)^\perp = J(D_2) \oplus \mu$ and F is a conformal semi-invariant submersion from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) , for $X \in \Gamma(D_2)$ and $Y \in \Gamma(\mu)$, we have

$$\frac{1}{\lambda^2} g_N(F_* JX, F_* Y) = g_M(JX, Y) = 0.$$

This implies that the distributions $F_*(JD_2)$ and $F_*(\mu)$ are orthogonal. Now, we investigate the geometry of the leaves of the distributitons D_1 and D_2 .

Theorem 3.6. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then D_1 defines a totally geodesic foliation on M if and only if*

$$(\nabla F_*)(X_1, JY_1) \in \Gamma(F_* \mu)$$

and

$$\frac{1}{\lambda^2} g_N((\nabla F_*)(X_1, JY_1), F_* CX) = g_M(Y_1, T_{X_1} \omega \mathcal{B}X)$$

for $X_1, Y_1 \in \Gamma(D_1)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. The distribution D_1 defines a totally geodesic foliation on M if and only if $g_M(\nabla_{X_1}^M Y_1, X_2) = 0$ and $g_M(\nabla_{X_1}^M Y_1, X) = 0$ for $X_1, Y_1 \in \Gamma(D_1)$, $X_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$. Using (2.1) and (2.2), we get

$$g_M(\nabla_{X_1}^M Y_1, X_2) = g_M(\mathcal{H} \nabla_{X_1}^M JY_1, JX_2).$$

Since F is a conformal semi-invariant submersion, using (2.10) we have

$$(3.19) \quad g_M(\nabla_{X_1}^M Y_1, X_2) = -\frac{1}{\lambda^2} g_N((\nabla F_*)(X_1, JY_1), F_* JX_2).$$

On the other hand, by using (2.1), (2.2), (2.6) and (3.4) we derive

$$g_M(\nabla_{X_1}^M Y_1, X) = g_M(Y_1, \nabla_{X_1}^M J\mathcal{B}X) + g_M(\mathcal{H} \nabla_{X_1}^M JY_1, \mathcal{C}X).$$

Now from (2.7), (2.10) and (3.3) we have

$$(3.20) \quad g_M(\nabla_{X_1}^M Y_1, X) = g_M(Y_1, T_{X_1} \omega \mathcal{B}X) - \frac{1}{\lambda^2} g_N((\nabla F_*)(X_1, JY_1), F_* CX).$$

Thus proof follows from (3.19) and (3.20). \square

For D_2 we have the following result.

Theorem 3.7. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then D_2 defines a totally geodesic foliation on M if and only if*

$$(\nabla F_*)(X_2, JX_1) \in \Gamma(F_* \mu)$$

and

$$-\frac{1}{\lambda^2} g_N(\nabla_{JY_2}^F F_* JX_2, F_* J\mathcal{C}X) = g_M(Y_2, \mathcal{B}T_{X_2} \mathcal{B}X) + g_M(X_2, Y_2) g_M(\mathcal{H} \text{grad} \ln \lambda, J\mathcal{C}X)$$

for $X_2, Y_2 \in \Gamma(D_2)$, $X_1 \in \Gamma(D_1)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. The distribution D_2 defines a totally geodesic foliation on M if and only if $g_M(\nabla_{X_2}^M Y_2, X_1) = 0$ and $g_M(\nabla_{X_2}^M Y_2, X) = 0$ for $X_2, Y_2 \in \Gamma(D_2)$, $X_1 \in \Gamma(D_1)$ and $X \in \Gamma((\ker F_*)^\perp)$. Since F is conformal semi-invariant submersion, using (2.1) and (2.2) and (2.10), we get

$$(3.21) \quad g_M(\nabla_{X_2}^M Y_2, X_1) = \frac{1}{\lambda^2} g_N((\nabla F_*)(X_2, JX_1), F_* JY_2).$$

On the other hand, by using (2.1), (2.2), (2.6) and (3.4) we derive

$$g_M(\nabla_{X_2}^M Y_2, X) = -g_M(JY_2, T_{X_2} \mathcal{B}X) + g_M([X_2, JY_2] + \nabla_{JY_2}^M X_2, \mathcal{C}X).$$

Since $[X_2, JY_2] \in \Gamma(\ker F_*)$, we obtain

$$g_M(\nabla_{X_2}^M Y_2, X) = -g_M(JY_2, T_{X_2} \mathcal{B}X) + g_M(\nabla_{JY_2}^M JX_2, JCX).$$

Then conformal semi-invariant submersion F , (3.4) and (2.10) imply

$$(3.22) \quad \begin{aligned} g_M(\nabla_{X_2}^M Y_2, X) &= g_M(Y_2, \mathcal{B}T_{X_2} \mathcal{B}X) + g_M(Y_2, X_2)g_M(\mathcal{H}grad \ln \lambda, JCX) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_{JY_2}^F F_* JX_2, F_* JCX). \end{aligned}$$

Thus proof follows from (3.21) and (3.22). \square

From Theorem 3.6 and Theorem 3.7, we have the following theorem;

Theorem 3.8. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) . Then the fibers of F are locally product manifold if and only if*

$$(\nabla F_*)(X_1, JY_1) \in \Gamma(F_* \mu),$$

$$\frac{1}{\lambda^2} g_N((\nabla F_*)(X_1, JY_1), F_* \mathcal{C}X) = g_M(Y_1, T_{X_1} \omega \mathcal{B}X)$$

and

$$(\nabla F_*)(X_2, JX_1) \in \Gamma(F_* \mu),$$

$$-\frac{1}{\lambda^2} g_N(\nabla_{JY_2}^F F_* JX_2, F_* JCX) = g_M(Y_2, \mathcal{B}T_{X_2} \mathcal{B}X) + g_M(X_2, Y_2)g_M(\mathcal{H}grad \ln \lambda, JCX)$$

for any $X_1, Y_1 \in \Gamma(D_1)$, $X_2, Y_2 \in \Gamma(D_2)$ and $X \in \Gamma((\ker F_*)^\perp)$.

4. HARMONICITY OF CONFORMAL SEMI-INVARIANT SUBMERSIONS

In this section, we are going to find necessary and sufficient conditions for a conformal semi-invariant submersions to be harmonic. We also investigate the necessary and sufficient conditions for such submersions to be totally geodesic. By considering the decomposition of the total space of conformal semi-invariant submersion, the following lemma comes from [2].

Lemma 4.1. *Let $F : (M^{2(m+n+r)}, g_M, J) \longrightarrow (N^{n+2r}, g_N)$ be a conformal semi-invariant submersion, where (M, g_M, J) is a Kähler manifold and (N, g_N) is a Riemannian manifold. Then the tension field τ of F is*

$$(4.1) \quad \tau(F) = -(2m+n)F_*(\mu^{\ker F_*}) + (2-n-2r)F_*(grad \ln \lambda),$$

where $\mu^{\ker F_*}$ is the mean curvature vector field of the distribution of $\ker F_*$.

From Lemma 4.1 we deduce that:

Theorem 4.1. *Let $F : (M^{2(m+n+r)}, g_M, J) \longrightarrow (N^{n+2r}, g_N)$ be a conformal semi-invariant submersion such that $n+2r \neq 2$ where (M, g_M, J) is a Kähler manifold and (N, g_N) is a Riemannian manifold. Then any three conditions below imply the fourth:*

- (i) F is harmonic
- (ii) The fibres are minimal
- (iii) F is a horizontally homothetic map.

We also have the following result.

Corollary 4.1. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If $n + 2r = 2$ then F is harmonic if and only if the fibres are minimal.*

Now we obtain necessary and sufficient condition for a conformal semi-invariant submersion to be totally geodesic. We recall that a differentiable map F between two Riemannian manifolds is called totally geodesic if

$$(\nabla F_*)(X, Y) = 0 \quad \forall X, Y \in \Gamma(TM).$$

A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. We now present the following definition.

Definition 4.1. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then F is called a (JD_2, μ) -totally geodesic map if*

$$(\nabla F_*)(JU, X) = 0, \text{ for } U \in \Gamma(D_2) \text{ and } X \in \Gamma((\ker F_*)^\perp).$$

In the sequel we show that this notion has an important effect on the geometry of the conformal submersion.

Theorem 4.2. *Let F be a conformal semi-invariant submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then F is a (JD_2, μ) -totally geodesic map if and only if F is horizontally homothetic map.*

Proof. For $U \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$, from Lemma 2.1, we have

$$(\nabla F_*)(JU, X) = JU(\ln \lambda)F_*X + X(\ln \lambda)F_*JU - g_M(JU, X)F_*(\text{grad} \ln \lambda).$$

From above equation, if F is a horizontally homothetic then $(\nabla F_*)(JU, X) = 0$. Conversely, if $(\nabla F_*)(JU, X) = 0$, we obtain

$$(4.2) \quad JU(\ln \lambda)F_*X + X(\ln \lambda)F_*JU = 0.$$

Taking inner product in (4.2) with F_*JU and since F is a conformal submersion, we write

$$g_M(\text{grad} \ln \lambda, JU)g_N(F_*X, F_*JU) + g_M(\text{grad} \ln \lambda, X)g_N(F_*JU, F_*JU) = 0.$$

Above equation implies that λ is a constant on $\Gamma(\mu)$. On the other hand, taking inner product in (4.2) with F_*X , we have

$$g_M(\text{grad} \ln \lambda, JU)g_N(F_*X, F_*X) + g_M(\text{grad} \ln \lambda, X)g_N(F_*JU, F_*X) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(JD_2)$. Thus λ is a constant on $\Gamma((\ker F_*)^\perp)$. Hence proof is complete. \square

Finally we give necessary and sufficient conditions for a conformal semi-invariant submersion to be totally geodesic.

Theorem 4.3. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant submersion, where (M, g_M, J) is a Kähler manifold and (N, g_N) is a Riemannian manifold. F totally geodesic map if and only if*

- (a) $CT_U JV + \omega \hat{\nabla}_U JV = 0 \quad U, V \in \Gamma(D_1)$
- (b) $\mathcal{CH} \nabla_U^M JW + \omega T_U JW = 0 \quad U \in \Gamma(\ker F_*), W \in \Gamma(D_2)$
- (c) F is a horizontally homothetic map.

Proof. (a) For $U, V \in \Gamma(D_1)$, using (2.2), (2.10) and (2.6) we have

$$(\nabla F_*)(U, V) = F_*(J(T_U JV + \hat{\nabla}_U JV)).$$

Using (3.3) and (3.4) in above equation we obtain

$$(\nabla F_*)(U, V) = F_*(\mathcal{B}T_U JV + CT_U JV + \phi \hat{\nabla}_U JV + \omega \hat{\nabla}_U JV).$$

Since $\mathcal{B}T_U JV + \phi \hat{\nabla}_U JV \in \Gamma(\ker F_*)$, we derive

$$(\nabla F_*)(U, V) = F_*(\mathcal{C}T_U JV + \omega \hat{\nabla}_U JV).$$

Then, since F is a linear isometry between $(\ker F_*)^\perp$ and TN , $(\nabla F_*)(U, V) = 0$ if and only if $\mathcal{C}T_U JV + \omega \hat{\nabla}_U JV = 0$.

(b) For $U \in \Gamma(\ker F_*)$, $W \in \Gamma(D_2)$, using (2.2) and (2.10) we have

$$\begin{aligned} (\nabla F_*)(U, W) &= \nabla_U^F F_* W - F_*(\nabla_U^M W) \\ &= F_*(J\nabla_U^M JW). \end{aligned}$$

Then from (2.7) we arrive at

$$(\nabla F_*)(U, W) = F_*(J(T_U JW + \mathcal{H}\nabla_U^M JW)).$$

Using (3.3) and (3.4) in above equation we obtain

$$(\nabla F_*)(U, W) = F_*(\phi T_U JW + \omega T_U JW + \mathcal{B}\mathcal{H}\nabla_U^M JW + \mathcal{C}\mathcal{H}\nabla_U^M JW).$$

Since $\phi T_U JW + \mathcal{B}\mathcal{H}\nabla_U^M JW \in \Gamma(\ker F_*)$, we derive

$$(\nabla F_*)(U, W) = F_*(\omega T_U JW + \mathcal{C}\mathcal{H}\nabla_U^M JW).$$

Then, since F is a linear isometry between $(\ker F_*)^\perp$ and TN , $(\nabla F_*)(U, W) = 0$ if and only if $\omega T_U JW + \mathcal{C}\mathcal{H}\nabla_U^M JW = 0$.

(c) For $X, Y \in \Gamma(\mu)$, from Lemma 2.1, we have

$$(\nabla F_*)(X, Y) = X(\ln \lambda)F_*Y + Y(\ln \lambda)F_*X - g_M(X, Y)F_*(\text{grad} \ln \lambda).$$

From above equation, taking $Y = JX$ for $X \in \Gamma(\mu)$ we obtain

$$\begin{aligned} (\nabla F_*)(X, JX) &= X(\ln \lambda)F_*JX + JX(\ln \lambda)F_*X - g_M(X, JX)F_*(\text{grad} \ln \lambda) \\ &= X(\ln \lambda)F_*JX + JX(\ln \lambda)F_*X. \end{aligned}$$

If $(\nabla F_*)(X, JX) = 0$, we obtain

$$(4.3) \quad X(\ln \lambda)F_*JX + JX(\ln \lambda)F_*X = 0.$$

Taking inner product in (4.3) with F_*X and since F is a conformal submersion, we write

$$g_M(\text{grad} \ln \lambda, X)g_N(F_*JX, F_*X) + g_M(\text{grad} \ln \lambda, JX)g_N(F_*X, F_*X) = 0.$$

Above equation implies that λ is a constant on $\Gamma(J\mu)$. On the other hand, taking inner product in (4.3) with F_*JX we have

$$g_M(\text{grad} \ln \lambda, X)g_N(F_*JX, F_*JX) + g_M(\text{grad} \ln \lambda, X)g_N(F_*X, F_*JX) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(\mu)$. In a similar way, for $U, V \in \Gamma(D_2)$, using Lemma 2.1 we have

$$(\nabla F_*)(JU, JV) = JU(\ln \lambda)F_*JV + JV(\ln \lambda)F_*JU - g_M(JU, JV)F_*(\text{grad} \ln \lambda).$$

From above equation, taking $V = U$ we obtain

$$\begin{aligned} (4.4) \quad (\nabla F_*)(JU, JU) &= JU(\ln \lambda)F_*JU + JU(\ln \lambda)F_*JU - g_M(JU, JU)F_*(\text{grad} \ln \lambda) \\ &= 2JU(\ln \lambda)F_*JU - g_M(JU, JU)F_*(\text{grad} \ln \lambda). \end{aligned}$$

Taking inner product in (4.4) with F_*JU and since F is a conformal submersion, we derive

$$2g_M(\text{grad} \ln \lambda, JU)g_N(F_*JU, F_*JU) - g_M(JU, JU)g_N(F_*(\text{grad} \ln \lambda), F_*JU) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(JD_2)$. Thus λ is a constant on $\Gamma((\ker F_*)^\perp)$. If F is a horizontally homothetic map, then $F_*(\text{grad} \ln \lambda)$ vanishes, thus the converse is clear, i.e. $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$. Hence proof is complete. \square

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